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Canonical transformations of local functionals and sh-Lie structures

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Abstract

In many Lagrangian field theories, there is a Poisson bracket on the space of local functionals. One may identify the fields of such theories as sections of a vector bundle. It is known that the Poisson bracket induces an sh-Lie structure on the graded space of horizontal forms on the jet bundle of the relevant vector bundle. We consider those automorphisms of the vector bundle which induce mappings on the space of functionals preserving the Poisson bracket and refer to such automorphisms as canonical automorphisms.

We determine how such automorphisms relate to the corresponding sh-Lie structure. If a Lie group acts on the bundle via canonical automorphisms, there are induced actions on the space of local functionals and consequently on the corresponding sh-Lie algebra. We determine conditions under which the sh-Lie structure induces an sh-Lie structure on a corresponding reduced space where the reduction is determined by the action of the group. These results are not directly a consequence of the corresponding theorems on Poisson manifolds as none of the algebraic structures are Poisson algebras.

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1. Introduction

The dynamical "equations of motion" of a Lagrangian field theory are usually derived from a variational principle of "least action". Given a Lagrangian L, the action of L is the functional S defined by

$$S(\phi) = \int_M L((j^n\phi)(x))Vol_M$$

where M is a manifold, ϕ may be either a vector-valued function or a section of a vector bundle E over M, and L is a real-valued function on some finite jet bundle J^nE. More generally, if $\pi : E \to M$ is a vector bundle and $\pi^\infty : J^\infty E \to M$ is the corresponding prolongation of E, then a smooth function $P : J^\infty E \to \mathbf{R}$ is called a *local function* on E provided that for some positive integer n there is a smooth function $P_n : J^nE \to \mathbf{R}$ such that $P = P_n \circ \pi_n$ where π_n is the projection of $J^\infty E$ onto J^nE. Thus all Lagrangians are local functions on an appropriate bundle. To say that $\mathcal{P}$ is a *local functional* on E means that $\mathcal{P}$ is a mapping from a subspace of compactly supported sections of $E \to M$ into $\mathbf{R}$ such that

$$\mathcal{P}(\phi) = \int_M (P \circ j^\infty\phi)(x)Vol_M$$

for some local function P and for all such sections ϕ of E.

In imitation of Hamiltonian mechanics one postulates the existence of a "Poisson bracket" on the space $\mathcal{F}$ of local functionals and then uses it to develop a Hamiltonian theory of fields. This bracket is assumed to satisfy the Jacobi identity and so defines a Lie algebra structure on the space $\mathcal{F}$. On the other hand there is no obvious commutative multiplication of such functionals and consequently $\mathcal{F}$ is not a Poisson algebra. This is such a well-known development that we may refer to standard monographs on the subject. In particular we call attention to [7] and [15] for classical expositions and to [8] for a quantum field theoretic development.

It was shown in [2] that a Poisson bracket on the space of local functionals induces what is known as sh-Lie structure on a part of the variational bicomplex which we refer to as the "de Rham complex" on $J^\infty E$. This sh-Lie structure is given by three mappings l_1, l_2, l_3 defined on this complex. The mapping l_2 is skew-symmetric and bilinear, and it may be regarded as defining a "bracket" but one which generally fails to satisfy the Jacobi identity. In fact l_2 satisfies the Jacobi identity if $l_3 = 0$. In a sense this sh-Lie structure is an anti-derivative of the Poisson bracket.

In the present paper we intend to develop ideas related to canonical transformations of these structures. Recall that if M is a Poisson manifold one says that a mapping from M to itself is a canonical transformation if and only if it preserves the Poisson bracket defined on $C^\infty(M)$ [14]. Moreover if one has a Lie group G which acts on M via canonical transformations, one obtains a reduction of the brackets to a reduced space M/G in the presence of appropriate hypothesis. The space of local functionals is not a Poisson algebra and so there is no underlying Poisson manifold. The bracket l_2 is defined on the space of "top" forms of the "de Rham complex" which can be identified with the space of local functions on $J^\infty E$. This space is a commutative algebra under pointwise multiplication, but l_2 does not satisfy the Jacobi identity and so again one does not have a Poisson manifold.

We say that an automorphism of the bundle E is *canonical* provided that its induced mapping on the space of local functionals, $\mathcal{F}$, preserves the Poisson bracket. We then determine how such automorphisms relate to the sh-Lie structure on $J^\infty E$. Finally we determine conditions under which there exists an sh-Lie structure on a reduced graded space in the presence of a Lie group which acts on E via canonical automorphisms.

We apply these ideas to a Poisson sigma model. Poisson sigma models have proven to be of interest in many areas of physics. In particular they have been used to describe certain two-dimensional theories of gravity by Ikeda [9], topological field theories by Schaller and Strobl [16], and to obtain a path integral proof of Kontsevich's theorem on deformation quantization by Cattaneo and Felder [6]. These are but a sample of the many authors who have made important contributions relating to these model theories.

After presenting some background material in Section 2, we find conditions for the *induced* automorphisms on the space of local functionals to be *canonical*. In Section 3 we show how these induced canonical transformations relate to the sh-Lie structure maps. In Section 4 we assume that one has a Lie group acting by canonical transformations on the space of functionals. We then determine how this action relates to the sh-Lie structure and find conditions for the existence of an (induced) sh-Lie structure on a corresponding reduced space. There is a brief discussion of functional invariance in Section 5. In Section 6 we show how our formalism applies to a specific Poisson sigma model due initially to Ikeda.

Clearly the questions dealt with in this paper relate more to the mathematical structures induced by a Poisson bracket on the space of functionals rather than to specific methods of solving dynamical field equations. Moreover we have restricted our attention to a class of theories in which the Poisson bracket is induced by a tensor ω which is scalar-valued rather than differential-operator valued. Once we understand this restricted case more fully we hope to extend these results to a larger class of theories for which ω is differential-operator valued.

Eventually we also intend to expand our scope to include fermionic theories such as those in [8]. Indeed, the sh-Lie formalism is particularly well-suited to interact with super-field theories such as those needed to describe the Batalin-Vilkovisky approach to BRST cohomology. Once anti-fields are introduced, our vector bundle E can be modified in such a manner that both the bosonic fields studied here and the fermionic (anti-)fields become sections of the new bundle. In this context the Batalin-Vilkovisky anti-bracket is none other than our Poisson bracket of local functionals with appropriate grading. Thus we expect the modifications of this work to the latter case to be minimal. In fact this work is motivated by both the classical field theories such as those described in [15] and the super-fields developed in [8]. This approach has proven its worth in investigations such as those found in [3] and [4].

2. Bundle automorphisms preserving the Poisson structure on the space of functionals

2.1. Background material

In this section we introduce some of the terminology and concepts that are used in this work, in addition to some of the simpler results that will be needed. Our exposition and notation closely follows that in [2]. First let $E \to M$ be a vector bundle where the base space M is an n-dimensional manifold and let $J^\infty E$ be the infinite jet bundle of E. The restriction of the infinite jet bundle over an appropriate open set $U \subset M$ is trivial with fiber an infinite dimensional vector space V^∞. The bundle

$$\pi^\infty : J^\infty E_U = U \times V^\infty \to U$$

then has induced coordinates given by

$$(x^i, u^a, u^a_i, u^a_{i_1 i_2}, \ldots).$$

We use multi-index notation and the summation convention throughout the paper. If $j^\infty\phi$ is the section of $J^\infty E$ induced by a section ϕ of the bundle E, then $u^a \circ j^\infty\phi = u^a \circ \phi$ and

$$u^a_I \circ j^\infty\phi = (\partial_{i_1}\partial_{i_2}, \ldots, \partial_{i_r})(u^a \circ j^\infty\phi)$$

where r is the order of the symmetric multi-index $I = \{i_1, i_2, \ldots, i_r\}$, with the convention that, for $r = 0$, there are no derivatives. For more details see [1] and [13].

Let Loc_E denote the algebra of local functions where a local function on $J^\infty E$ is defined to be the pull-back of a smooth function on some finite jet bundle $J^p E$ via the projection from $J^\infty E$ to $J^p E$. Let Loc^0_E denote the subalgebra of Loc_E such that $P \in Loc^0_E$ iff $(j^\infty\phi)^* P$ has compact support for all $\phi \in \Gamma E$ with compact support, where ΓE denotes the set of sections of the bundle $E \to M$. The de Rham complex of differential forms $\Omega^*(J^\infty E, d)$ on $J^\infty E$ possesses a differential ideal, the ideal C of contact forms θ which satisfy $(j^\infty\phi)^*\theta = 0$ for all sections ϕ with compact support. This ideal is generated by the contact one-forms, which in local coordinates assume the form $\theta^a_J = \mathrm{d}u^a_J - u^a_{iJ}\mathrm{d}x^i$.

Using the contact forms, we see that the complex $\Omega^*(J^\infty E, d)$ splits as a bicomplex $\Omega^{r,s}(J^\infty E)$ (though the finite level complexes $\Omega^*(J^p E)$ do not), where $\Omega^{r,s}(J^\infty E)$ denotes the space of differential forms on $J^\infty E$ with r horizontal components and s vertical components. The bigrading is described by writing a differential p-form $\alpha = \alpha^{\mathbf{J}}_{IA}(\theta^A_{\mathbf{J}} \wedge \mathrm{d}x^I)$ as an element of $\Omega^{r,s}(J^\infty E)$, with $p = r + s$, and

$$\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r}, \qquad \theta^A_{\mathbf{J}} = \theta^{a_1}_{J_1} \wedge \ldots \wedge \theta^{a_s}_{J_s}.$$

Now let C_0 denote the set of contact one-forms of *order zero*. Contact one-forms of order zero satisfy $(j^1\phi)^*(\theta) = 0$ and, in local coordinates, they assume the form $\theta^a = \mathrm{d}u^a - u^a_i \mathrm{d}x^i$. Notice that both C_0 and $\Omega^{n,1} = \Omega^{n,1}(J^\infty E)$ are modules over Loc_E. Let $\Omega^{n,1}_0$ denote the subspace of $\Omega^{n,1}$ which is locally generated by the forms $\{(\theta^a \wedge \mathrm{d}^n x)\}$ over Loc_E. We assume the existence of a mapping, ω, from $\Omega^{n,1}_0 \times \Omega^{n,1}_0$ to Loc_E, such that ω is a skew-symmetric module homomorphism in each variable separately. In local coordinates let $\omega^{ab} = \omega(\theta^a \wedge \nu, \theta^b \wedge \nu)$, where ν is a volume element on M (notice that in local coordinates ν takes the form $\nu = f\mathrm{d}^n x = f\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n$ for some function $f : U \to \mathbf{R}$ and U is a subset of M on which the x^i's are defined).

We will assume throughout this paper that ω satisfies the conditions that make our Poisson bracket, which will be defined soon, satisfy the Jacobi identity.

Define the operator D_i (total derivative) by $D_i = \partial/\partial x^i + u^a_{iJ}(\partial/\partial u^a_J)$ (recall we assume the summation convention, i.e., the sum is over all a and multi-index J), and recall that the Euler-Lagrange operator maps $\Omega^{n,0}(J^\infty E)$ into $\Omega^{n,1}(J^\infty E)$ and is defined by

$$\mathbf{E}(P\nu) = \mathbf{E}_a(P)(\theta^a \wedge \nu)$$

where $P \in Loc_E$, ν is a volume form on the base manifold M, and the components $\mathbf{E}_a(P)$ are given by

$$\mathbf{E}_a(P) = (-D)_I \left(\frac{\partial P}{\partial u^a_I}\right).$$

For simplicity of notation we may use $\mathbf{E}(P)$ for $\mathbf{E}(P\nu)$. We will also use $\tilde{D}_i$ for $\partial/\partial\tilde{x}^i + \tilde{u}^a_{iJ}(\partial/\partial\tilde{u}^a_J)$ and $\tilde{\mathbf{E}}_a(P)$ for $(-\tilde{D})_I(\partial P/\partial\tilde{u}^a_I)$ so that $\mathbf{E}(P) = \tilde{\mathbf{E}}_a(P)(\tilde{\theta}^a \wedge \nu)$ in the $(\tilde{x}^\mu, \tilde{u}^a)$ coordinate system.

Let $\Omega^{k,l}_c(J^\infty E)$ be the subspace of $\Omega^{k,l}(J^\infty E)$, for $\{k,l\} \neq \{n,0\}$, such that $\alpha \in \Omega^{k,l}_c(J^\infty E)$ iff $(j^\infty\phi)^*\alpha$ has compact support for all $\phi \in \Gamma E$ with compact support, and let $\Omega^{n,0}_c(J^\infty E)$ be the subspace of $\Omega^{n,0}(J^\infty E)$ such that $P\nu \in \Omega^{n,0}_c(J^\infty E)$ iff $(j^\infty\phi)^*(P\nu)$ and $(j^\infty\phi)^*\mathbf{E}_a(P)$ have compact support for all $\phi \in \Gamma E$ with compact support and for all a. We are interested in the complex

$$0 \to \Omega^{0,0}_c(J^\infty E) \to \Omega^{1,0}_c(J^\infty E) \to \cdots \to \Omega^{n-1,0}_c(J^\infty E) \to \Omega^{n,0}_c(J^\infty E)$$

with the differential d_H defined by $\mathrm{d}_H = \mathrm{d}x^i D_i$, i.e., if $\alpha = \alpha_I \mathrm{d}x^I$ then $\mathrm{d}_H\alpha = D_i\alpha_I \mathrm{d}x^i \wedge \mathrm{d}x^I$. Notice that this complex is exact whenever the base manifold M is contractible (e.g. see [5]).

Now let $\mathcal{F}$ be the space of functionals where $\mathcal{P} \in \mathcal{F}$ iff $\mathcal{P} = \int_M P\nu$ for some $P \in Loc^0_E$, and define a Poisson bracket on $\mathcal{F}$ by

$$\{\mathcal{P}, \mathcal{Q}\}(\phi) = \int_M [\omega(\mathbf{E}(P), \mathbf{E}(Q)) \circ j\phi]\nu,$$

where $\phi \in \Gamma E$, ν is a volume form on M, $\mathcal{P} = \int_M P\nu$, $\mathcal{Q} = \int_M Q\nu$, and $P, Q \in Loc^0_E$. Using local coordinates (x^μ, u^a_I) on $J^\infty E$, observe that for $\phi \in \Gamma E$ such that the support of ϕ lies in the domain Ω of some chart x of M, one has

$$\{\mathcal{P}, \mathcal{Q}\}(\phi) = \int_{x(\Omega)} ([\omega^{ab}\mathbf{E}_a(P)\mathbf{E}_b(Q)] \circ j\phi \circ x^{-1})(x^{-1})^*(\nu)$$

where x^{-1} is the inverse of $x = (x^\mu)$.

We assume that ω satisfies the necessary conditions for the above bracket to satisfy the Jacobi identity, e.g. see [15]. *Notice that it follows from the identity (7.11) of* [15] *that the bracket satisfies the Jacobi identity if the skew-symmetric matrix $\{\omega^{ab}\}$ is a Poisson tensor* *in the sense that:*

$$\omega^{cd}\frac{\partial\omega^{ab}}{\partial u^d}+\omega^{ad}\frac{\partial\omega^{bc}}{\partial u^d}+\omega^{bd}\frac{\partial\omega^{ca}}{\partial u^d}=0,$$

where $\{u^a\}$ *are coordinates on the fiber of the trivial bundle E. This condition is met in the case of the Poisson sigma model, which we include later in the paper, and more generally for any trivial vector bundle with a Poisson structure on its fibers.*

The functions P and Q in our definition of the Poisson bracket of local functionals are representatives of $\mathcal{P}$ and $\mathcal{Q}$ respectively, since generally these are not unique. In fact $\mathcal{F}\simeq H^n_c(J^\infty E)$, where $H^n_c(J^\infty E)=\Omega^{n,0}_c(J^\infty E)/(im\mathrm{d}_H\bigcap\Omega^{n,0}_c(J^\infty E))$ and imd_H is the image of the differential d_H.

Let $\psi:E\to E$ be an automorphism, sending fibers to fibers, and let $\psi_M:M\to M$ be the induced diffeomorphism of M. Notice that ψ induces an automorphism $j\psi:J^\infty E\to J^\infty E$ where

$$(j\psi)((j^\infty\phi)(p))=j(\psi\circ\phi\circ\psi_M^{-1})(\psi_M(p)),$$

for all $\phi\in\Gamma E$ and all p in the domain of ϕ. In these coordinates the independent variables transform via $\tilde{x}^\mu=\psi^\mu_M(x^\nu)$. Local coordinate representatives of ψ_M and $j\psi$ may be described in terms of charts (Ω,x) and $(\tilde{\Omega},\tilde{x})$ of M, and induced charts $((\pi^\infty)^{-1}(\Omega),(x^\mu,u^a_I))$ and $((\pi^\infty)^{-1}(\tilde{\Omega}),(\tilde{x}^\mu,\tilde{u}^a_I))$ of $J^\infty E$.

Remark. In Section 4 we will consider (left) Lie group actions on E and their induced (left) actions on $J^\infty E$. Such actions are defined by homomorphisms from the group into the group of automorphisms of E.

Definition. $\omega:\Omega^{n,1}_0\times\Omega^{n,1}_0\to Loc_E$ is *covariant* with respect to an automorphism $\psi:E\to E$ of the above form iff

$$\omega((j\psi)^*\theta,(j\psi)^*\theta')=(\det\psi_M)(j\psi)^*(\omega(\theta,\theta')),$$

for all $\theta,\theta'\in\Omega^{n,1}_0(J^\infty E)$.

Observe that

$$\begin{aligned}(j\psi)^*\tilde{\theta}^a&=(j\psi)^*(\mathrm{d}\tilde{u}^a-\tilde{u}^a_\mu\mathrm{d}\tilde{x}^\mu)\\&=\mathrm{d}(\tilde{u}^a\circ j\psi)-(\tilde{u}^a_\mu\circ j\psi)\mathrm{d}(\tilde{x}^\mu\circ j\psi)\\&=\frac{\partial\psi^a_E}{\partial x^\nu}\mathrm{d}x^\nu+\frac{\partial\psi^a_E}{\partial u^b}\mathrm{d}u^b-\left(\frac{\partial\psi^a_E}{\partial x^\nu}+\frac{\partial\psi^a_E}{\partial u^b}u^b_\nu\right)(J^{-1})^\nu_\mu\frac{\partial\tilde{x}^\mu\circ j\psi}{\partial x^\lambda}\mathrm{d}x^\lambda\\&=\frac{\partial\psi^a_E}{\partial x^\nu}\mathrm{d}x^\nu+\frac{\partial\psi^a_E}{\partial u^b}\mathrm{d}u^b-\frac{\partial\psi^a_E}{\partial x^\nu}\mathrm{d}x^\nu-\frac{\partial\psi^a_E}{\partial u^b}u^b_\nu\mathrm{d}x^\nu\\&=\frac{\partial\psi^a_E}{\partial u^b}(\mathrm{d}u^b-u^b_\nu\mathrm{d}x^\nu)\\&=\frac{\partial\psi^a_E}{\partial u^b}\theta^b\end{aligned}$$

where we have assumed that $\psi_E^a = \tilde{u}^a \circ \psi$ and J is the Jacobian matrix of the transformation $\tilde{x}^\nu = \psi_M^\nu(x^\mu)$.

Lemma 2.1. *The following are equivalent*

(i) $\omega((j\psi)^*\theta, (j\psi)^*\theta') = (\det\psi_M)(j\psi)^*(\omega(\theta, \theta'))$, *for all* $\theta, \theta' \in \Omega_0^{n,1}(J^\infty E)$.

(ii) $\tilde{\omega}^{ab} \circ j\psi = (\det\psi_M)\omega^{cd} \frac{\partial\psi_E^a}{\partial u^c} \frac{\partial\psi_E^b}{\partial u^d}$.

Proof. Notice that

$$\begin{aligned}\det\psi_M(j\psi)^*(\omega(\mathbf{E}(P), \mathbf{E}(Q))) &= (\det\psi_M)\omega(\mathbf{E}(P), \mathbf{E}(Q)) \circ j\psi \\ &= \det\psi_M[\omega(\tilde{\theta}^a \wedge \nu, \tilde{\theta}^b \wedge \nu)\tilde{\mathbf{E}}_a(P)\tilde{\mathbf{E}}_b(Q)] \circ j\psi \\ &= \det\psi_M(\tilde{\omega}^{ab} \circ j\psi)(\tilde{\mathbf{E}}_a(P) \circ j\psi)(\tilde{\mathbf{E}}_b(Q) \circ j\psi)\end{aligned}$$

and that

$$\begin{aligned}(j\psi)^*(\mathbf{E}(P)) &= (j\psi)^*(\tilde{\mathbf{E}}_a(P)(\tilde{\theta}^a \wedge \nu)) \\ &= (\tilde{\mathbf{E}}_a(P) \circ j\psi)\frac{\partial\psi_E^a}{\partial u^c}(\det\psi_M)(\theta^c \wedge \nu).\end{aligned}$$

Now

$$\begin{aligned}\omega((j\psi)^*\mathbf{E}(P), (j\psi)^*\mathbf{E}(Q)) &= (\det\psi_M)^2\frac{\partial\psi_E^a}{\partial u^c}\frac{\partial\psi_E^b}{\partial u^d}(\tilde{\mathbf{E}}_a(P) \circ j\psi)(\tilde{\mathbf{E}}_b(Q) \circ j\psi) \\ &\quad \times \omega(\theta^c \wedge \nu, \theta^d \wedge \nu) \\ &= (\det\psi_M)^2\frac{\partial\psi_E^a}{\partial u^c}\frac{\partial\psi_E^b}{\partial u^d}\omega^{cd}(\tilde{\mathbf{E}}_a(P) \circ j\psi)(\tilde{\mathbf{E}}_b(Q) \circ j\psi).\end{aligned}$$

Hence $\omega((j\psi)^*\mathbf{E}(P), (j\psi)^*\mathbf{E}(Q)) = (\det\psi_M)(j\psi)^*(\omega(\mathbf{E}(P), \mathbf{E}(Q)))$ for all P, Q in Loc_E iff $(\tilde{\omega}^{ab} \circ j\psi) = (\det\psi_M)\partial\psi_E^a/\partial u^c \times \partial\psi_E^b/\partial u^d \times \omega^{cd}$.

2.2. Automorphisms preserving the Poisson structure

Let $L : J^\infty E \to \mathbf{R}$ be a Lagrangian in Loc_E (generally we will assume that any element of Loc_E is a Lagrangian). Let $\hat{L} = L \circ (x^\mu, u_I^a)^{-1}$ and let $\tilde{L} = L \circ (\tilde{x}^\mu, \tilde{u}_I^a)^{-1}$. Then, in local coordinates, $\tilde{L}$ is related to $\hat{L}$ by the equation

$$(\tilde{L} \circ j\overline{\psi})\det(J) = \hat{L},$$

where $j\overline{\psi} = (\tilde{x}^\nu, \tilde{u}_K^b) \circ (x^\mu, u_I^a)^{-1}$ and J is the Jacobian matrix of the transformation $\psi_M = \tilde{x}^\nu \circ (x^\mu)^{-1}$. With abuse of notation we may assume coordinates and charts are the same and write $\tilde{x}^\nu = \psi_M(x^\mu)$. For simplicity, we have also assumed that ψ_M is orientation-preserving.

In this case the functional

$$\tilde{\mathcal{L}} = \int_{\tilde{\Omega}} \tilde{L} \mathrm{d}^n \tilde{x}$$

is the transformed form of the functional

$$\hat{\mathcal{L}} = \int_{\Omega} \hat{L} \mathrm{d}^n x$$

where $\hat{L}$ and $\tilde{L}$ are related as above, Ω is the domain of integration and $\tilde{\Omega}$ is the transformed domain under $j\overline{\psi}$ (see [15], pp. 249–250). Notice that both of these are local coordinate expressions of the equation $\mathcal{L} = \int_M L\nu$, for appropriately restricted charts. Now suppose that ψ is an automorphism of E, $j\psi$ its induced automorphism on $J^\infty E$, and ψ_M its induced (orientation-preserving) diffeomorphism on M. Also suppose that $\hat{L}$ and $\tilde{L}$ are two Lagrangians related by the equation $(\tilde{L} \circ j\psi)\det(\psi_M) = \hat{L}$. We have:

Lemma 2.2. *Let P be a Lagrangian as above, then*

$$\mathbf{E}_a((P \circ j\psi)\det(\psi_M)) = \det(\psi_M)\frac{\partial \psi_E^c}{\partial u^a}(\tilde{\mathbf{E}}_c(P) \circ j\psi). \tag{2.1}$$

Proof. First notice that $\mathbf{E}_{u^a}(\hat{L}) = \det(\psi_M)\partial\psi_E^c/\partial u^a(\mathbf{E}_{\tilde{u}^c}(\tilde{L}) \circ j\psi)$ (see [15], p. 250). But $(\tilde{L} \circ j\psi)\det(\psi_M) = \hat{L}$. The identity (2.1) follows by letting $P = \tilde{L}$. Notice that this is justified since $\tilde{L}$ is arbitrary in the sense that given any L' there exists an $\hat{L}$ derived from a Lagrangian L as above such that $(L' \circ j\psi)\det(\psi_M) = \hat{L}$ since $j\psi$ is an automorphism.

Let $\hat{\psi}$ denote the mapping representing the induced action of the automorphism on sections of E, i.e., $\hat{\psi} : \Gamma E \to \Gamma E$ where $\hat{\psi}(\phi) = \psi \circ \phi \circ \psi_M^{-1}$ and ϕ is a section of E. This induces a mapping on the space of local functionals given by

$$\begin{aligned}(\mathcal{P} \circ \hat{\psi})(\phi) &= \mathcal{P}(\psi \circ \phi \circ \psi_M^{-1}) \\ &= \int_M [P \circ j(\psi \circ \phi \circ \psi_M^{-1})]\nu \\ &= \int_M [P \circ j(\psi \circ j\phi \circ \psi_M^{-1})]\nu \\ &= \int_M [P \circ j\psi \circ j\phi](\det\psi_M)\nu,\end{aligned}$$

where

$$\mathcal{P}(\phi) = \int_M (P \circ j\phi)\nu,$$

and ϕ is a section of E.

We find conditions on those automorphisms of the space of functionals under which the Poisson structure is preserved.

Recall that $\{\mathcal{P}, \mathcal{Q}\} = \int_M \omega(\mathbf{E}(P), \mathbf{E}(Q))\nu$, and hence

$$\{\mathcal{P}, \mathcal{Q}\}(\phi) = \int_M [\omega(\mathbf{E}(P), \mathbf{E}(Q)) \circ j\phi]\nu. \tag{2.2}$$

Now

$$\{\mathcal{P} \circ \hat{\psi}, \mathcal{Q} \circ \hat{\psi}\}(\phi) = (\{\mathcal{P}, \mathcal{Q}\} \circ \hat{\psi})(\phi) \tag{2.3}$$

is equivalent to

$$\int_M [\omega(\mathbf{E}((P \circ j\psi)\mathrm{det}\psi_M), \mathbf{E}((Q \circ j\psi)\mathrm{det}\psi_M)) \circ j\phi]\nu$$
$$= \int_M [(\omega(\mathbf{E}(P), \mathbf{E}(Q)) \circ j\psi \circ j\phi)\mathrm{det}\psi_M]\nu,$$

but since this holds for all sections ϕ of E it is equivalent to

$$\omega(\mathbf{E}((P \circ j\psi)\mathrm{det}\psi_M), \mathbf{E}((Q \circ j\psi)\mathrm{det}\psi_M)) = (\omega(\mathbf{E}(P), \mathbf{E}(Q)) \circ j\psi)\mathrm{det}\psi_M$$

up to a *divergence*. The last equation is equivalent to

$$\omega^{ab}\mathbf{E}_b((Q \circ j\psi)\mathrm{det}\psi_M)\mathbf{E}_a((P \circ j\psi)\mathrm{det}\psi_M) = ([\tilde{\omega}^{ab}\tilde{\mathbf{E}}_b(Q)\tilde{\mathbf{E}}_a(P)] \circ j\psi)\mathrm{det}\psi_M$$

up to a divergence, or (Lemma 2.2)

$$\omega^{ab}(\mathrm{det}\psi_M)^2 \frac{\partial\psi_E^d}{\partial u^b}\frac{\partial\psi_E^c}{\partial u^a}(\tilde{\mathbf{E}}_d(Q) \circ j\psi)(\tilde{\mathbf{E}}_c(P) \circ j\psi)$$
$$= (\tilde{\omega}^{ab} \circ j\psi)(\tilde{\mathbf{E}}_b(Q) \circ j\psi)(\tilde{\mathbf{E}}_a(P) \circ j\psi)(\mathrm{det}\psi_M)$$

up to a divergence. Finally, since the last equation is true for all P and Q it is equivalent to

$$\tilde{\omega}^{ab} \circ j\psi = (\mathrm{det}\psi_M)\omega^{cd}\frac{\partial\psi_E^a}{\partial u^c}\frac{\partial\psi_E^b}{\partial u^d}$$

which is equivalent to the covariance of ω. (Notice that if the last equality does not hold then by some choice of P and Q the equations above will not hold up to a divergence.) We have established the following:

Theorem 2.3. *Let $\psi : E \to E$ be an automorphism of E sending fibers to fibers, and let $\Psi : \mathcal{F} \to \mathcal{F}$ be the induced mapping defined by $\Psi(\mathcal{P}) = \mathcal{P} \circ \hat{\psi}$ (noting that $\mathcal{P} \circ \hat{\psi}$ is defined as above) where $\hat{\psi} : \Gamma E \to \Gamma E$ is given by $\hat{\psi}(\phi) = \psi \circ \phi \circ \psi_M^{-1}$. Then Ψ is canonical in the sense that*

$$\{\Psi(\mathcal{P}), \Psi(\mathcal{Q})\} = \Psi(\{\mathcal{P}, \mathcal{Q}\})$$

for all $\mathcal{P}, \mathcal{Q} \in \mathcal{F}$ iff ω is covariant with respect to ψ.

Definition. An automorphism ψ of E is *canonical* provided the induced mapping $\Psi : \mathcal{F} \longrightarrow \mathcal{F}$ is canonical (in the sense of the preceding theorem).

Example. Consider $M = \mathbf{R}$, $E = \mathbf{R} \times \mathbf{R}^2$, and let

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider the action $\psi_g : E \to E$ defined by $\psi_g(x, u^1, u^2) = (x, g \cdot (u^1, u^2))$ for some $g \in SO(2)$. It can easily be shown that ω is covariant with respect to ψ_g and hence the induced action is canonical. As an illustration let g be the counter-clockwise rotation by 90^o (so that $gu^1 = u^2$ and $gu^2 = -u^1$), and let $\mathcal{P} = \int_{\mathbf{R}} P(u^1)\mathrm{d}x$ and $\mathcal{Q} = \int_{\mathbf{R}} Q(u^2)\mathrm{d}x$ for real-valued differentiable functions P and Q. Then $\Psi(\mathcal{P}) = \int_{\mathbf{R}} P(u^2)\mathrm{d}x$, $\Psi(\mathcal{Q}) = \int_{\mathbf{R}} Q(-u^1)\mathrm{d}x$ and $\{\Psi(\mathcal{P}), \Psi(\mathcal{Q})\} = -\int_{\mathbf{R}} P'(u^2)Q'(-u^1)\mathrm{d}x$. On the other hand $\{\mathcal{P}, \mathcal{Q}\} = -\int_{\mathbf{R}} P'(u^1)Q'(u^2)\mathrm{d}x$, so that $\Psi(\{\mathcal{P}, \mathcal{Q}\}) = -\int_{\mathbf{R}} P'(u^2)Q'(-u^1)\mathrm{d}x$.

The following will be needed in our subsequent work.

Proposition 2.4. $\mathrm{d}_H((j\psi)^*\eta) = (j\psi)^*(\mathrm{d}_H\eta)$ *for* $\eta \in \Omega^{m,0}$, *m arbitrary.*

Proof. Let $\eta = \alpha_I d\tilde{x}^I$ where each $\alpha_I : J^\infty E \to \mathbf{R}$. We assume that the α_I's depend on the transformed variables $(\tilde{x}^j, \tilde{u}^b_K)$. Then

$$\begin{aligned}
\mathrm{d}_H((j\psi)^*(\alpha)) &= \mathrm{d}_H((\alpha_I \circ j\psi)\mathrm{d}(\tilde{x}^I \circ \psi)) \\
&= \{D_i(\alpha_I \circ j\psi)\}\mathrm{d}x^i \wedge \mathrm{d}(\tilde{x}^I \circ \psi) \\
&= \left\{\left(\frac{\partial \alpha_I}{\partial \tilde{x}^j} \circ j\psi\right)\frac{\partial(\tilde{x}^j \circ \psi)}{\partial x^i} + \left(\frac{\partial \alpha_I}{\partial \tilde{u}^b_K} \circ j\psi\right)\frac{\partial j\psi^b_K}{\partial x^i} + u^a_{iI}\left(\frac{\partial \alpha_I}{\partial \tilde{u}^b_K} \circ j\psi\right)\frac{\partial j\psi^b_K}{\partial u^a_I}\right\} \\
&\quad \times \mathrm{d}x^i \wedge \mathrm{d}(\tilde{x}^I \circ \psi) \\
&= \left\{\left(\frac{\partial \alpha_I}{\partial \tilde{x}^j} \circ j\psi\right)\mathrm{d}(\tilde{x}^j \circ \psi) + \left(\frac{\partial \alpha_I}{\partial \tilde{u}^b_K} \circ j\psi\right)\left(\frac{\partial j\psi^b_K}{\partial x^i} + u^a_{iI}\frac{\partial j\psi^b_K}{\partial u^a_I}\right)\mathrm{d}x^i\right\} \\
&\quad \times \wedge \mathrm{d}(\tilde{x}^I \circ \psi) \\
&= \left\{\left(\frac{\partial \alpha_I}{\partial \tilde{x}^j} \circ j\psi\right) + \left(\frac{\partial \alpha_I}{\partial \tilde{u}^b_K} \circ j\psi\right)\left(\frac{\partial j\psi^b_K}{\partial x^i} + u^a_{iI}\frac{\partial j\psi^b_K}{\partial u^a_I}\right)(J^{-1})^i_j\right\}\mathrm{d}(\tilde{x}^j \circ \psi) \\
&\quad \times \wedge \mathrm{d}(\tilde{x}^I \circ \psi) \\
&= \left\{\left(\frac{\partial \alpha_I}{\partial \tilde{x}^j} \circ j\psi\right) + \left(\frac{\partial \alpha_I}{\partial \tilde{u}^b_K} \circ j\psi\right)(\tilde{u}^b_{Kj} \circ j\psi)\right\}\mathrm{d}(\tilde{x}^j \circ \psi) \wedge \mathrm{d}(\tilde{x}^I \circ \psi) \\
&= \{(\tilde{D}_j\alpha_I) \circ j\psi\}(\mathrm{d}(\tilde{x}^j \circ \psi) \wedge \mathrm{d}(\tilde{x}^I \circ \psi)) = (j\psi)^*(\mathrm{d}_H(\alpha)),
\end{aligned}$$

where we assumed that $j\psi^b_K = \tilde{u}^b_K \circ j\psi$ and J is the Jacobian matrix of the transformation $\psi^\nu_M(x^\mu)$ as before. □

3. Canonical automorphisms and sh-Lie algebras

In this section we consider the structure maps of the sh-Lie algebra on the horizontal complex $\{\Omega^{i,0}\}$. Throughout the remainder of the paper we assume the horizontal complex is exact. A complete description of these maps can be found in [2] or one of the references therein; however it is useful to give a brief overview.

3.1. Overview of sh-Lie algebras

Let $\mathcal{F}$ be a vector space and (X_*, l_1) a homological resolution thereof, i.e., X_* is a graded vector space, l_1 is a differential and lowers the grading by one with $\mathcal{F} \simeq H_0(l_1)$ and $H_k(l_1) = 0$ for $k > 0$. The complex (X_*, l_1) is called the resolution space. (We are *not* using the term 'resolution' in a categorical sense.) Consider a homological resolution of the space $\mathcal{F}$ of local functionals as in [2]. In the field theoretic framework considered in [2] it was shown that under certain hypothesis (see the theorem below) the Lie structure defined by the Poisson bracket on $\mathcal{F}$ induces an sh-Lie structure on the graded vector space $X_i = \Omega^{n-i,0}(J^\infty E)$, for $0 \le i < n$ and $X_n = \Omega^{0,0}(J^\infty E)$. For completeness we give the definition of sh-Lie algebras and include a statement of the relevant theorem.

Definition. An sh-Lie structure on a graded vector space X_* is a collection of linear, skew-symmetric maps $l_k : \bigotimes^k X_* \to X_*$ of degree $k-2$ that satisfy the relation

$$\sum_{i+j=n+1} \sum_{unsh(i,n-i)} e(\sigma)(-1)^\sigma (-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), \dots, x_{\sigma(n)}) = 0,$$

where $1 \le i, j$.

Notice that in this definition $e(\sigma)$ is the Koszul sign which depends on the permutation σ as well as on the degree of the elements $x_1, x_2, \dots, x_n$ (see for example [10]).

Remark. Although this may seem to generally be a rather complicated structure, it simplifies drastically in the case of field theory where, aside from the differential $l_1 = \mathrm{d}_H$, the only non-zero maps are l_2 and l_3 in degree 0.

The theorem relevant to field theory depends on the existence of a linear skew-symmetric map $\tilde{l}_2 : X_0 \otimes X_0 \to X_0$ (in our case the the Poisson bracket will be the integral of this mapping as we will see in detail shortly) satisfying conditions (i) and (ii) below. These conditions are all that is needed in order that an sh-Lie structure exists.

Theorem 3.1. *A skew-symmetric linear map $\tilde{l}_2 : X_0 \otimes X_0 \to X_0$ that satisfies conditions (i) and (ii) below extends to an sh-Lie structure on the resolution space X_*;*

(i) $\tilde{l}_2(c, b_1) = b_2$
(ii) $\sum_{\sigma \in unsh(2,1)} (-1)^\sigma \tilde{l}_2(\tilde{l}_2(c_{\sigma(1)}, c_{\sigma(2)}), c_{\sigma(3)}) = b_3$

where c, c_1, c_2, c_3 are cycles and b_1, b_2, b_3 are boundaries in X_0.

Subsequently we will suppress some of the notation and assume the summands are over the appropriate shuffles with their corresponding signs.

We also assume the existence of a chain homotopy s which satisfies $-s \circ l_1 = 1 + l_1 \circ s$. Using this chain homotopy one can define l_3 in degree 0 by the composition $s \circ \tilde{l}_2 \circ \tilde{l}_2$ which we may write simply as $s\tilde{l}_2\tilde{l}_2$ (we assume the sum over the three unshuffles for $\tilde{l}_2\tilde{l}_2$ with their corresponding signs).

3.2. *The effect of a canonical automorphism on the structure maps of the sh-Lie algebra*

To apply the theorem in the last subsection, we need a candidate for $\tilde{l}_2$. We may define such a mapping on $\Omega^{n,0}$ by

$$\tilde{l}_2(P\nu, Q\nu) = \omega^{ab}\mathbf{E}_a(P)\mathbf{E}_b(Q)\nu = \omega(\mathbf{E}(P), \mathbf{E}(Q))\nu. \tag{3.4}$$

Recall that for each automorphism ψ we have $(j\psi)^*(P\nu) = (P \circ j\psi)(\det\psi_M)\nu$. Therefore

$$\tilde{l}_2((j\psi)^*(P\nu), (j\psi)^*(Q\nu)) = (j\psi)^*(\tilde{l}_2(P\nu, Q\nu)) \tag{3.5}$$

for all $P, Q \in Loc_E$ if and only if

$$\omega^{ab}\mathbf{E}_b((Q \circ j\psi)\det\psi_M)\mathbf{E}_a((P \circ j\psi)\det\psi_M) = [(\tilde{\omega}^{ab}\tilde{\mathbf{E}}_b(Q)\tilde{\mathbf{E}}_a(P)) \circ j\psi](\det\psi_M),$$

for all $P, Q \in Loc_E$, which by Lemma 2.2 is equivalent to

$$\omega^{ab}(\det\psi_M)^2\frac{\partial\psi^d}{\partial u^b}\frac{\partial\psi^c}{\partial u^a}(\tilde{\mathbf{E}}_d(Q) \circ j\psi)(\tilde{\mathbf{E}}_c(P) \circ j\psi)$$
$$= (\tilde{\omega}^{ab} \circ j\psi)(\tilde{\mathbf{E}}_b(Q) \circ j\psi)(\tilde{\mathbf{E}}_a(P) \circ j\psi)(\det\psi_M).$$

The last equation is true for all $P, Q \in Loc_E$, so it is equivalent to

$$\tilde{\omega}^{ab} \circ j\psi = (\det\psi_M)\omega^{cd}\frac{\partial\psi_E^a}{\partial u^c}\frac{\partial\psi_E^b}{\partial u^d}$$

which in turn is equivalent to the covariance of ω.

Now consider l_3 in degree 0. We have

$$\begin{aligned}
l_3((j\psi)^*(P\nu), (j\psi)^*(Q\nu), (j\psi)^*(R\nu)) &= s[\tilde{l}_2(\tilde{l}_2((j\psi)^*(P\nu), (j\psi)^*(Q\nu)), (j\psi)^*(R\nu))] \\
&= s[\tilde{l}_2((j\psi)^*(\tilde{l}_2(P\nu, Q\nu)), (j\psi)^*(R\nu))] \\
&= s[(j\psi)^*(\tilde{l}_2(\tilde{l}_2(P\nu, Q\nu), R\nu))] \\
&= s[-(j\psi)^*(l_1 l_3(P\nu, Q\nu, R\nu))] \\
&= s[-l_1((j\psi)^* l_3(P\nu, Q\nu, R\nu))]
\end{aligned}$$

since in this case $l_1 = \mathrm{d}_H$ so it commutes with the pull-back (Proposition 2.4). Proceeding using the identity

$$-s \circ l_1 = 1 + l_1 \circ s$$

the above becomes

$$(1 + l_1 \circ s)[(j\psi)^*(l_3(P\nu, Q\nu, R\nu))] = (j\psi)^*(l_3(P\nu, Q\nu, R\nu)) + l_1 \circ s[(j\psi)^*(l_3(P\nu, Q\nu, R\nu))].$$

So $l_3((j\psi)^*(P\nu), (j\psi)^*(Q\nu), (j\psi)^*(R\nu)) = (j\psi)^*(l_3(P\nu, Q\nu, R\nu))$ up to an exact form. We have shown:

Theorem 3.2. *Let $\psi : E \to E$ be an automorphism of E sending fibers to fibers, and let $j\psi : J^\infty E \to J^\infty E$ be its induced automorphism on $J^\infty E$. Then*

$$\tilde{l}_2((j\psi)^*\alpha, (j\psi)^*\beta) = (j\psi)^*(\tilde{l}_2(\alpha, \beta))$$

for all $\alpha, \beta \in \Omega^{n,0}(J^\infty E)$ iff ω is covariant with respect to ψ. Moreover we then have

$$l_3((j\psi)^*\alpha, (j\psi)^*\beta, (j\psi)^*\gamma) = (j\psi)^* l_3(\alpha, \beta, \gamma) + l_1(\delta),$$

for all $\alpha, \beta, \gamma \in \Omega^{n,0}(J^\infty E)$, and for some $\delta \in \Omega^{n-2,0}(J^\infty E)$.

4. Reduction of the graded vector space in field theory

Let M be a manifold, $E \to M$ a vector bundle, and $J^\infty E$ the infinite jet bundle of E. Let G be a Lie group acting on E via automorphisms (as in Section 2) and hence inducing an action of G on $J^\infty E$. We assume the induced action $\hat{\psi}_g$ on ΓE is canonical with respect to the Poisson bracket of local functionals for all $g \in G$. Notice that G acts via canonical tranformations on the space of functionals iff for every $j\psi_g$

$$\tilde{l}_2((j\psi_g)^* f_1, (j\psi_g)^* f_2) = (j\psi_g)^*(\tilde{l}_2(f_1, f_2)),$$

where $\tilde{l}_2$ is defined on the vector space $\Omega^{n,0}(J^\infty E)$ as in the previous section (in fact $\tilde{l}_2(f_1, f_2) = 1/2[\omega(\mathbf{E}(f_1), \mathbf{E}(f_2)) - \omega(\mathbf{E}(f_2), \mathbf{E}(f_1))]$, see also Eq. (3.4)).

Definition. Given an automorphism ψ of the bundle E, a differential form $\alpha \in \Omega^{k,l}(J^\infty E)$ is ψ-*invariant* iff $(j\psi)^*\alpha = \alpha$. If G acts on E via automorphisms $\psi_g : E \to E$, $g \in G$, then α is G-*invariant* iff it is ψ_g-invariant for all $g \in G$.

Let $\Omega^{k,l}_\psi(J^\infty E)$ denote the space of all ψ-invariant forms on $J^\infty E$ which are in $\Omega^{k,l}_c(J^\infty E)$, and let $\Omega^{k,l}_G(J^\infty E)$ denote the space of all G-invariant forms in $\Omega^{k,l}_c(J^\infty E)$.

One also needs the following:

Definition. Assume that G acts on E such that $J^\infty E/G$ has a manifold structure and the canonical projection map $\pi : J^\infty E \to J^\infty E/G$ is smooth. Then $\Omega_c^{k,0}(J^\infty E/G)$ is the subspace of k-forms $\alpha \in \Omega_c^k(J^\infty E/G)$ such that $\pi^*\alpha \in \Omega_G^{k,0}(J^\infty E)$, and $\Omega_c^{*,0}(J^\infty E/G)$ is the *reduced graded space* of the graded space $\Omega_c^{*,0}(J^\infty E)$ with respect to G.

We are interested in actions that send fibers to fibers, i.e. the transformation of the independent variables does not depend on the dependent variables, so that acting on an element of $\Omega_c^{k,0}(J^\infty E)$ *gives an element of the same space and the reduction to* $\Omega_c^{*,0}(J^\infty E/G)$ *makes sense.*

In fact we will also assume that the map ψ_M representing the transformation of the independent variables ($x^\nu = \psi_M^\nu(x^\mu)$) is the identity for all $g \in G$ (see the proposition that follows). This will enable us to define a differential on the reduced graded space. It will also insure that the space $\Omega_c^{n,0}(J^\infty E)$ does not collapse to zero upon reduction (due to a reduction in the number of independent variables so that any n-form in $\Omega_c^{n,0}(J^\infty E/G)$ would be trivial) which is desired so that the induced sh-Lie structure would not necessarily be trivial.

Proposition 4.1. *If* ψ_M *is the identity map, then* $\pi^* : \Omega_c^{k,0}(J^\infty E/G) \to \Omega_G^{k,0}(J^\infty E)$ *is onto.*

Proof. Notice that if ψ_M is the identity map one can choose coordinates $\{x^i\}$ on $J^\infty E/G$ such that $\pi^* x^i = x^i$, $i = 1, 2, \ldots, n$ (where by an abuse of notation we denote by x^i both coordinates on M and $J^\infty E/G$) are the coordinates on M. Now $\pi^*(\alpha_I \mathrm{d}x^I) = (\alpha_I \circ \pi)\mathrm{d}x^I$ where $|I| = k$, i.e. we are assuming $\alpha_I \mathrm{d}x^I \in \Omega_c^{k,0}(J^\infty E/G)$. Since π is a smooth canonical projection, it is clear that for any smooth G-invariant function f on $J^\infty E$ there exists a smooth function α_I on $J^\infty E/G$ such that $f = \alpha_I \circ \pi$. The result follows. □

Corollary 4.2. *If* ψ_M *is the identity map, then we have an isomorphism* $\pi^* : \Omega_c^{k,0}(J^\infty E/G) \longrightarrow \Omega_G^{k,0}(J^\infty E)$.

In this setting it can be shown that $\Omega_c^{*,0}(J^\infty E/G)$ is a complex with a differential $\hat{\mathrm{d}}_H : \Omega_c^{m,0}(J^\infty E/G) \longrightarrow \Omega_c^{m+1,0}(J^\infty E/G)$ defined by

$$\hat{\mathrm{d}}_H h = (\pi^*)^{-1}(\mathrm{d}_H(\pi^* h)).$$

This is well defined since $\mathrm{d}_H(\pi^* h)$ is invariant under the group action which follows from the fact that $\pi^* h$ is invariant under the group action so that $(j\psi_g)^*(\mathrm{d}_H(\pi^* h)) = \mathrm{d}_H((j\psi_g)^*(\pi^* h)) = \mathrm{d}_H(\pi^* h)$. Also notice that $\hat{\mathrm{d}}_H \circ \hat{\mathrm{d}}_H = 0$ easily follows from $\mathrm{d}_H \circ \mathrm{d}_H = 0$. So $\hat{\mathrm{d}}_H$ is a well-defined differential.

Reduction hypothesis. Assume that every invariant d_H-exact form is the horizontal differential of an invariant form. This hypothesis will guarantee that the reduced graded space with the differential $\hat{\mathrm{d}}_H$ is exact. Subsequently we will determine sufficient conditions which will insure that this is true.

This assumption will also yield the two conditions, (i) and (ii) below, that are needed to obtain the sh-Lie structure on the reduced graded space.

Lemma 4.3. *Suppose that* $\Omega_c^{*,0}(J^\infty E)$ *is exact. If for every* d_H*-exact form* $\alpha \in \Omega_G^{k,0}(J^\infty E)$ *there exists* $\gamma \in \Omega_G^{k-1,0}(J^\infty E)$ *such that* $\alpha = \mathrm{d}_H \gamma$*, then the reduced graded space is exact.*

Proof. Suppose that $\hat{\mathrm{d}}_H\beta = 0$, then $\pi^*(\hat{\mathrm{d}}_H\beta) = 0$ and by the definition of $\hat{\mathrm{d}}_H$ this implies that $\mathrm{d}_H(\pi^*\beta) = 0$. Now exactness of $\Omega_c^{*,0}$ implies that $\pi^*\beta = \mathrm{d}_H\gamma$ for some γ, and γ can be chosen so that it is invariant by assumption since $\mathrm{d}_H\gamma$ is, so $\gamma = \pi^*\tau$ for some τ. By the definition of $\hat{\mathrm{d}}_H$ then $\mathrm{d}_H(\pi^*\tau) = \pi^*(\hat{\mathrm{d}}_H\tau)$ so that $\pi^*\beta = \pi^*(\hat{\mathrm{d}}_H\tau)$ or

$$\pi^*(\beta - \hat{\mathrm{d}}_H\tau) = 0$$

from which $\beta - \hat{\mathrm{d}}_H\tau = 0$, and therefore $\beta = \hat{\mathrm{d}}_H\tau$. (*Observe that* $\pi^*\delta = \delta \circ \mathrm{d}\pi = 0$ *implies that* $\delta = 0$ *for* $\delta \in \Omega_c^{k,0}(J^\infty E/G)$ *since* $\mathrm{d}\pi$ *is onto*.) □

Remark. We have used the simplified notation $\Omega_c^{*,0}$ for $\Omega_c^{*,0}(J^\infty E)$.

Corollary 4.4. *Under the same hypothesis as in the preceeding lemma, the subcomplex of G-invariant forms,* $\Omega_G^{*,0}(J^\infty E)$, *is exact.*

Now we proceed to finding a map on the reduced space $\Omega_c^{n,0}(J^\infty E/G)$ analogous to and induced by $\tilde{l}_2$ on the space $\Omega_c^{n,0}(J^\infty E)$. Define $\hat{l}_2$ by

$$(\hat{l}_2(f_1, f_2)) = (\pi^*)^{-1}\tilde{l}_2(\pi^* f_1, \pi^* f_2)$$

where $f_1, f_2 \in \Omega_c^{n,0}(J^\infty E/G)$. Notice that this is well-defined since $\tilde{l}_2(\pi^* f, \pi^* h)$ is invariant under the group action by the following calculation

$$(j\psi_g)^*\tilde{l}_2(\pi^* f, \pi^* h) = \tilde{l}_2((j\psi_g)^*(\pi^* f), (j\psi_g)^*(\pi^* h)) = \tilde{l}_2(\pi^* f, \pi^* h),$$

and the map $(\pi^*)^{-1}$ exists by Corollary 4.2.

Skew-symmetry and linearity of $\hat{l}_2$ follow from the skew-symmetry and linearity of $\tilde{l}_2$. Furthermore $\hat{l}_2$ satisfies

(i) $\hat{l}_2(\hat{\mathrm{d}}_H k_1, h) = \hat{\mathrm{d}}_H k_2$,
(ii) $\sum_{\sigma\in unsh(2,1)}(-1)^\sigma \hat{l}_2(\hat{l}_2(f_{\sigma(1)}, f_{\sigma(2)}), f_{\sigma(3)}) = \hat{\mathrm{d}}_H k_3$,

where $k_1 \in \Omega_c^{n-1,0}(J^\infty E/G)$, while $h, f_1, f_2, f_3 \in \Omega_c^{n,0}(J^\infty E/G)$, and for some $k_2, k_3 \in \Omega_c^{n-1,0}(J^\infty E/G)$. Recall that we may suppress some of the notation and assume the summands are over the appropriate shuffles with their corresponding signs.

To verify (i) notice that

$$\begin{aligned}\pi^*(\hat{l}_2(\hat{\mathrm{d}}_H k_1, h)) &= \tilde{l}_2(\pi^*(\hat{\mathrm{d}}_H k_1), \pi^* h)\\ &= \tilde{l}_2(\mathrm{d}_H(\pi^* k_1), \pi^* h)\\ &= \mathrm{d}_H K_2,\end{aligned}$$

for some $K_2 \in \Omega_c^{n-1,0}(J^\infty E)$. But by our assumption K_2 can be chosen to be invariant under the group action since $\mathrm{d}_H K_2$ is, i.e., $K_2 = \pi^* k_2$ for some $k_2 \in \Omega_c^{n-1,0}(J^\infty E/G)$, and then $\mathrm{d}_H K_2 = \mathrm{d}_H(\pi^* k_2) = \pi^*(\hat{\mathrm{d}}_H k_2)$ by the definition of $\hat{\mathrm{d}}_H$. This implies that $\hat{l}_2(\hat{\mathrm{d}}_H k_1, h) =$

$\hat{\mathrm{d}}_H k_2$. (*Recall that* $\pi^*\alpha = \alpha \circ \mathrm{d}\pi = 0$ *implies that* $\alpha = 0$ *for* $\alpha \in \Omega_c^{n,0}(J^\infty E/G)$ *since* $\mathrm{d}\pi$ *is onto*.)

While to verify (ii), notice that

$$\pi^*(\hat{l}_2(\hat{l}_2(f_1, f_2), f_3)) = \tilde{l}_2(\tilde{l}_2(\pi^* f_1, \pi^* f_2), \pi^* f_3) = \mathrm{d}_H K_3,$$

where the sum is over the unshuffles (2,1), and for some $K_3 \in \Omega_c^{n-1,0}(J^\infty E)$ and all $f_1, f_2, f_3 \in \Omega_c^{n,0}(J^\infty E/G)$. Again K_3 can be chosen to be invariant under the group action since $\mathrm{d}_H K_3$ is, i.e., $K_3 = \pi^* k_3$ for some $k_3 \in \Omega_c^{n-1,0}(J^\infty E/G)$, and then $\mathrm{d}_H K_3 = \mathrm{d}_H(\pi^* k_3) = \pi^*(\hat{\mathrm{d}}_H k_3)$ by the definition of $\hat{\mathrm{d}}_H$. This implies that $\hat{l}_2(\hat{l}_2(f_1, f_2), f_3) = \hat{\mathrm{d}}_H k_3$.

We have shown (see Lemmas 1 and 2 in [2] and Theorem 3.1):

Theorem 4.5. *There exists a skew-symmetric bilinear bracket on* $H_0(\hat{\mathrm{d}}_H) \times H_0(\hat{\mathrm{d}}_H)$ *that satisfies the Jacobi identity, where we are using* $H_0(\hat{\mathrm{d}}_H)$ *for* $H^n(\Omega_c^{*,0}(J^\infty E/G), \hat{\mathrm{d}}_H)$. *This bracket is induced by the map* $\hat{l}_2$.

Theorem 4.6. The skew-symmetric linear map $\hat{l}_2$ *as defined above on the space* $\Omega_c^{n,0}(J^\infty E/G)$ *extends to an sh-Lie structure on the graded space* $\Omega_c^{*,0}(J^\infty E/G)$.

4.1. Exactness of the reduced graded space

In this section we find sufficient conditions under which our reduction hypothesis in the last section holds. Thus we consider the question: If α is in the reduced space $\Omega_c^{k-1,0}(J^\infty E/G)$ and $\hat{\mathrm{d}}_H\alpha = 0$, then is $\alpha = \hat{\mathrm{d}}_H\beta$ for some β? Suppose $\hat{\mathrm{d}}_H\alpha = 0$ for $\alpha \in \Omega_c^{k,0}(J^\infty E/G)$, then $\mathrm{d}_H(\pi^*\alpha) = 0$ so that $\pi^*\alpha = \mathrm{d}_H\gamma$ for some $\gamma \in \Omega_c^{k-1,0}(J^\infty E)$ since $\Omega_c^{*,0}(J^\infty E)$ is exact. Notice that $\mathrm{d}_H\gamma$ is invariant under the group action (since $\mathrm{d}_H\gamma = \pi^*\alpha$) so $\mathrm{d}_H\gamma = j\psi_g^*(\mathrm{d}_H\gamma)$ for all $g \in G$, or since d_H commutes with $j\psi_g^*$ by Proposition 2.4, $\mathrm{d}_H\gamma = \mathrm{d}_H(j\psi_g^*\gamma)$ for all $g \in G$. So

$$\gamma = j\psi_g^*\gamma + \mathrm{d}_H\tau_g,$$

where $\tau_g \in \Omega_c^{k-2,0}(J^\infty E)$ depends on g. Consider $\gamma' = \gamma + \mathrm{d}_H\Delta$ for some fixed $\Delta \in \Omega_c^{k-2,0}(J^\infty E)$ and notice that $\mathrm{d}_H\gamma' = \mathrm{d}_H\gamma = \pi^*\alpha$. Now $\gamma' = \gamma + \mathrm{d}_H\Delta = j\psi_g^*\gamma + \mathrm{d}_H\tau_g + \mathrm{d}_H\Delta$ so that $j\psi_g^*\gamma = \gamma' - \mathrm{d}_H\tau_g - \mathrm{d}_H\Delta$, and hence $j\psi_g^*\gamma' = j\psi_g^*\gamma + j\psi_g^*(\mathrm{d}_H\Delta) = \gamma' - \mathrm{d}_H\tau_g - \mathrm{d}_H\Delta + j\psi_g^*(\mathrm{d}_H\Delta)$. But if γ' is invariant under the group action then $-\mathrm{d}_H\Delta - \mathrm{d}_H\tau_g + j\psi_g^*(\mathrm{d}_H\Delta) = 0$ or

$$\mathrm{d}_H(j\psi_g^*\Delta - \Delta - \tau_g) = 0,$$

(recall that d_H commutes with $j\psi_g^*$ by Proposition 2.4) so that $(j\psi_g^*\Delta - \Delta - \tau_g)$ is exact. Notice that this is a necessary and sufficient condition for the exactness of the reduced space. In this case let $\beta = \pi_*\gamma'$ and notice that $\pi^*(\hat{\mathrm{d}}_H\beta) = \mathrm{d}_H\gamma' = \pi^*\alpha$ so that $\hat{\mathrm{d}}_H\beta = \alpha$.

Observe that τ_g depends on g and on γ whereas Δ depends on γ.

We find the above criterion too general and rather complicated, and find it useful to consider a special case. Suppose that G is *compact* and let $\alpha \in \Omega_c^{k,0}(J^\infty E)$ be a closed form that is invariant under the group action. By exactness of $\Omega^{*,0}$ there exists a β such that $\mathrm{d}_H\beta = \alpha$. Observe that $\mathrm{d}_H(j\psi_g^*\beta) = j\psi_g^*(\mathrm{d}_H\beta) = j\psi_g^*\alpha = \alpha$ for all $g \in G$. So

$$\int_G \mathrm{d}_H(j\psi_g^*\beta)\mathrm{d}g = \int_G \alpha \mathrm{d}g = \alpha \int_G \mathrm{d}g = \alpha \cdot vol(G) = \alpha$$

assuming that $vol(G) = 1$. Now let

$$\hat{\beta} = \int_G (j\psi_g^*\beta)\mathrm{d}g$$

and notice that $\mathrm{d}_H\hat{\beta} = \int_G \mathrm{d}_H(j\psi_g^*\beta)\mathrm{d}g = \alpha$, and $j\psi_h^*(\hat{\beta}) = \int_G j\psi_h^*(j\psi_g^*\beta)\mathrm{d}g = \int_G (j\psi_{gh}^*\beta)\mathrm{d}(gh) = \hat{\beta}$. So we have:

Proposition 4.7. *If the group G acting (canonically) on E is compact, then every* d_H*-closed form that is G-invariant is the horizontal differential of a G-invariant form. Consequently* $\Omega_c^{*,0}(J^\infty E/G)$ *is exact and admits an (induced) sh-Lie structure.*

4.2. The existence of an sh-Lie structure on the subcomplex of G-invariant forms

In this subsection we consider the subcomplex of G-invariant forms

$$\cdots \to \Omega_G^{n-1,0}(J^\infty E) \overset{\mathrm{d}_H}{\to} \Omega_G^{n,0}(J^\infty E).$$

Working with the subcomplex of G-invariant forms is rather interesting. We shall maintain the same assumptions made earlier in this section, in particular the hypotheses of Lemma 4.3. Recall that throughout this section we require the mapping ψ_M representing the tranformation of the independent variables to be the identity.

In fact, if the base manifold M is one-dimensional these assumptions are not needed for this subcomplex to be exact. However their absence does not guarantee the existence of an sh-Lie structure (that's obtained from the original one).

Recall that by Corollary 4.4 the subcomplex of G-invariant forms is exact and observe that $\tilde{l}_2(\alpha, \beta) = \tilde{l}_2((j\psi)^*\alpha, (j\psi)^*\beta) = (j\psi)^*\tilde{l}_2(\alpha, \beta)$ for $\alpha, \beta \in \Omega_G^{n,0}(J^\infty E)$. So $\tilde{l}_2$ can be restricted to the subspace $\Omega_G^{n,0}(J^\infty E)$, and if we combine this with exactness, we notice that conditions (i) and (ii) that guarantee the existence of the sh-Lie structure, as stated earlier in this section, are readily established (this sh-Lie structure is just the restriction of the original one to the subcomplex of G-invariant forms). So we have:

Theorem 4.8. *Under the same hypotheses as in* Lemma 4.3, *there exists an sh-Lie structure on the subcomplex of G-invariant forms* $\Omega_G^{*,0}(J^\infty E)$.

Example. Consider $M = \mathbf{R}$, $E = \mathbf{R} \times \mathbf{R}^2$, and let

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider the action of $G = SO(2)$ on E defined by $\psi_g(x, u^1, u^2) = (x, g \cdot (u^1, u^2))$, $g \in SO(2)$, and observe that $SO(2)$ is compact. As we noticed in the example in Section 1, ω is covariant with respect to ψ_g and hence the induced action is canonical for all $g \in G$. Consider a subset of $J^\infty E$ defined by $(J^\infty E)' = J^\infty E - \{u \in J^\infty E | u = (u_I^1, u_I^2) = (0, 0), I = 0, 1, 2, 3, \ldots\}$, where $u_0^1 = u^1, u_1^1 = u_x^1, u_2^1 = u_{xx}^1, \ldots$ etc.

Notice that at a point in $(J^\infty E)'$ u^1 and u^2 cannot be zero at the same time, u_x^1 and u_x^2 cannot be zero at the same time, and so on. We note that $(J^\infty E)'/G = \mathbf{R} \times (\mathbf{R}^+) \times (\mathbf{R}^+) \times \mathbf{S^1} \times (\mathbf{R}^+) \times \mathbf{S^1} \times \cdots$, where $\mathbf{R}^+$ is the set of all positive real numbers (without the 0). As an illustration let $\alpha = u^1 \mathrm{d}x$ and notice that $(j\psi_g)^*\alpha = (\cos\theta)u^1 + (\sin\theta)u^2$, whereas $\beta = (1/2)[(u^1)^2 + (u^2)^2]\mathrm{d}x$ is invariant under the induced action of G and so is $\gamma = (1/2)[(u_x^1)^2 + (u_x^2)^2]\mathrm{d}x$. As for the sh-Lie structure, first notice that the resolution space is rather simple:

$$0 \to Loc_E^0 \to \Omega_c^{1,0}(J^\infty E).$$

One finds that

$$\tilde{l}_2(P\mathrm{d}x, Q\mathrm{d}x) = (\mathbf{E}_1(P)\mathbf{E}_2(Q) - \mathbf{E}_2(P)\mathbf{E}_1(Q))\mathrm{d}x,$$

while $l_2(P\mathrm{d}x, f) = 0$ for $f \in Loc_E^0$ since $l_2 l_1(P\mathrm{d}x, f) = l_2(l_1(P\mathrm{d}x), f) + l_2(P\mathrm{d}x, l_1 f) = 0 + l_2(P\mathrm{d}x, \mathrm{d}_H f) = 0$ which follows from $\mathbf{E}_i(\mathrm{d}_H f) = 0$, $i = 1, 2$. Further $l_2 = 0$ in higher degrees. We note that l_3 is non-zero on degree 0, but is zero on higher degrees. Let for example $P_1 = u^1 u_x^2$, $P_2 = u^1 u^2$, and $P_3 = (u^1)^2$, then $\tilde{l}_2\tilde{l}_2(P_1\mathrm{d}x, P_2\mathrm{d}x, P_3\mathrm{d}x) = 4u^1 u_x^1 \mathrm{d}x = \mathrm{d}_H(2(u^1)^2)$, so that we can choose $l_3(P_1\mathrm{d}x, P_2\mathrm{d}x, P_3\mathrm{d}x) = -2(u^1)^2$.

The subcomplex of G-invariant forms is exact and so the sh-Lie structure can be restricted to it. Observe that, for example, $\tilde{l}_2(\beta, \gamma) = \tilde{l}_2((1/2)[(u^1)^2 + (u^2)^2]\mathrm{d}x, (1/2)[(u_x^1)^2 + (u_x^2)^2]\mathrm{d}x) = (-u^1 u_{xx}^2 + u^2 u_{xx}^1)\mathrm{d}x$ is invariant under the induced group action of $G = SO(2)$.

Remark. In [12] Kogan and Olver provide a definitive account of invariant Euler-Lagrange equations using moving frames and other tools from differential geometry. We note that their invariantization map ι in our case will just map horizontal differentials to themselves (i.e. in local coordinates $\iota(\mathrm{d}x^i) = \mathrm{d}x^i$). Consequently the invariant derivatives are the same as the ordinary derivatives $\mathcal{D}_i = D_i$ (here we are borrowing some of the notation from [12]) whereas the twisted invariant adjoints of $\mathcal{D}_i$ turn out to be $\mathcal{D}_i^\dagger = -\mathcal{D}_i$. Now suppose that $I^1, \ldots, I^m$ is a fundamental set of differential invariants (on $J^\infty E$). Let $\hat{I}^1, \ldots, \hat{I}^m$ be coordinates on $J^\infty E/G$ such that $\hat{I}^k \circ \pi = I^k$, $k = 1, \ldots, m$ where π is the canonical projection map $\pi : J^\infty E \to J^\infty E/G$, and let $\hat{L}$ be the corresponding Lagrangian defined on $J^\infty E/G$ for some (invariant) Lagrangian $L = \hat{L} \circ \pi$ defined on $J^\infty E$. The *invariant*

Eulerian (in [12]) is defined by

$$\mathcal{E}_\alpha(\tilde{L}) = \sum_K \mathcal{D}^\dagger_K \frac{\partial \tilde{L}}{\partial I^\alpha_K}$$

where $\tilde{L}$ indicates the Lagrangian is written in terms of the I^α_K's. But this $\mathcal{E}_\alpha$, in our case, reduces to

$$\mathcal{E}_\alpha(\tilde{L}) = \sum_K (-D)_K \frac{\partial \tilde{L}}{\partial I^\alpha_K}.$$

A *corresponding invariant Eulerian* can be defined on $J^\infty E/G$. To accomplish this, first define "total derivative" $\hat{D}_k$ on $J^\infty E/G$ by $(\hat{D}_k \hat{P}) \circ \pi = D_k(\hat{P} \circ \pi)$ where $\hat{P}$ is a smooth function on $J^\infty E/G$. Now let $\hat{I}^\alpha_K = \hat{D}_K \hat{I}^\alpha$ and notice that $\hat{I}^\alpha_K \circ \pi = I^\alpha_K = D_K I^\alpha$. Finally, define

$$\hat{\mathcal{E}}_\alpha(\hat{L}) = \sum_K (-\hat{\mathcal{D}})_K \frac{\partial \hat{L}}{\partial \hat{I}^\alpha_K}$$

and observe that $\hat{\mathcal{E}}_\alpha(\hat{L}) \circ \pi = \mathcal{E}_\alpha(\tilde{L})$. The reader should consult [12] for more details.

5. Functional invariance

In this section we consider the implications of invariance on the space of functionals. Assume that ψ is canonical, i.e.

$$\{\mathcal{P} \circ \hat{\psi}, \mathcal{Q} \circ \hat{\psi}\} = \{\mathcal{P}, \mathcal{Q}\} \circ \hat{\psi}$$

for all functionals $\mathcal{P}$, $\mathcal{Q}$.

Definition. We say that $\mathcal{P}$ is invariant under ψ iff $\mathcal{P} \circ \hat{\psi} = \mathcal{P}$.

Notice that this holds if, and only if

$$\int_M (P \circ j\psi \circ j\phi) \det\psi_M \nu = \int_M (P \circ j\phi)\nu$$

for all $\phi \in \Gamma E$, which in turn holds if $(P \circ j\psi)\det\psi_M = P$. Similarly one says that $P \in Loc^0_E$ is invariant under ψ iff $(P \circ j\psi)\det\psi_M = P$. Let $\mathcal{F}_\psi$ denote the set of all functionals $\mathcal{P}$ such that $\mathcal{P} \circ \hat{\psi} = \mathcal{P}$. Observe that

$$\mathcal{P}, \mathcal{Q} \in \mathcal{F}_\psi \Rightarrow \{\mathcal{P}, \mathcal{Q}\} \in \mathcal{F}_\psi$$

so $\mathcal{F}_\psi$ is a Lie subalgebra of $\mathcal{F}$. Let $Loc_E^0(\psi)$ denote the subset of Loc_E^0 consisting of $P \in Loc_E^0$ such that

$$P = (P \circ j\psi)\det\psi_M.$$

We note that $Loc_E^0(\psi)$ is a subspace of Loc_E^0, while for automorphisms ψ such that $\det\psi_M = 1$, $Loc_E^0(\psi)$ is a subalgebra of Loc_E^0.

Proposition 5.1. *If $P\nu \in \Omega^{n,0}(J^\infty E)$ is ψ-invariant for an automorphism ψ, then so is* $\mathbf{E}(P\nu)$.

Proof. In local coordinates $\mathbf{E}(P\nu) = \mathbf{E}_a(P)(\theta^a \wedge \nu)$. So

$$\begin{aligned}(j\psi)^*(\mathbf{E}(P\nu)) &= (j\psi)^*(\mathbf{E}_a(P))(j\psi)^*(\theta^a \wedge \nu)\\ &= (\mathbf{E}_a(P) \circ j\psi)\left(\frac{\partial \psi_E^a}{\partial u^b}\theta^b \wedge (\det\psi_M)\nu\right)\\ &= (\mathbf{E}_a(P) \circ j\psi)\frac{\partial \psi_E^a}{\partial u^b}(\det\psi_M)(\theta^b \wedge \nu).\end{aligned}$$

Now, since $P\nu$ is ψ-invariant we have

$$\begin{aligned}P\nu &= (j\psi)^*(P\nu)\\ &= (j\psi)^*(P)(j\psi)^*\nu\\ &= (P \circ j\psi)(\det\psi_M)\nu,\end{aligned}$$

and therefore $(P \circ j\psi)\det\psi_M = P$. Finally,

$$\begin{aligned}\mathbf{E}(P\nu) &= \mathbf{E}_a(P)(\theta^a \wedge \nu)\\ &= \mathbf{E}_a((P \circ j\psi)\det\psi_M)(\theta^a \wedge \nu)\\ &= (\det\psi_M)\frac{\partial \psi_E^b}{\partial u^a}(\mathbf{E}_b(P) \circ j\psi)(\theta^a \wedge \nu)\\ &= (j\psi)^*(\mathbf{E}(P\nu)),\end{aligned}$$

where we have used Lemma 2.2 in the last calculation. □

If G is a Lie group which acts on E via canonical automorphisms ψ_g, for all $g \in G$, then we write

$$Loc_E^0(G) = \bigcap_{g\in G} Loc_E^0(\psi_g), \qquad \mathcal{F}_G = \bigcap_{g\in G} \mathcal{F}_{\psi_g}.$$

Clearly $\mathcal{F}_G$ is a Lie sub-algebra of $\mathcal{F}$, and if $P \in Loc_E^0(G)$, then $\mathbf{E}(P\nu)$ is G-invariant.

Notice that if ϕ is a section of the bundle $E \to M$ then $j^\infty\phi$ is a section of $\pi^\infty : J^\infty E \to M$. Sections of this type are said to be *holonomic* as they are induced by a section of $E \to M$. It is easily shown that not all sections of π^∞ are holonomic. Observe that $\pi \circ j^\infty\phi$ is a section of the bundle $\tau : J^\infty E/G \to M$ since $\pi^\infty = \tau \circ \pi$. Similarly we say that a section η of τ is *holonomic* if it has the form $\eta = \pi \circ j^\infty\phi$ for some section $j^\infty\phi$ of π^∞. Let Γ denote the set of all holonomic sections of τ. Note that Γ is not a linear space over **R** since π is not linear. Indeed $J^\infty E/G$ is generally not a vector bundle.

Definition. We say that $\tilde{\mathcal{P}}$ is a *reduced local functional* if it is a mapping from the set Γ of holonomic sections of the bundle $\tau : J^\infty E/G \to M$ into **R** such that

$$\tilde{\mathcal{P}}(\eta) = \int_M \eta^*(\tilde{P})\nu$$

for some smooth mapping $\tilde{P} : J^\infty E/G \to \mathbf{R}$ and for every $\eta \in \Gamma$. We denote the set of all reduced local functionals by $\tilde{\mathcal{F}}$.

In this definition, when we say that $\tilde{P} : J^\infty E/G \to \mathbf{R}$ is smooth we mean that $\tilde{P} \circ \pi$ is in Loc_E^0.

Proposition 5.2. *There is a bijection Ξ from $\tilde{\mathcal{F}}$ onto $\mathcal{F}_G$. The mapping Ξ is defined as follows: if $\tilde{\mathcal{P}}$ is defined by*

$$\tilde{\mathcal{P}}(\eta) = \int_M \eta^*(\tilde{P})\nu$$

for some smooth mapping $\tilde{P}$, then $\Xi(\tilde{\mathcal{P}}) = \mathcal{P}$ is defined by

$$\mathcal{P}(\phi) = \int_M (j^\infty\phi)^*(P)\nu$$

where $P = \tilde{P} \circ \pi$.

The proof of the proposition is straightforward and is omitted.

Remark. It follows from the proposition that the set $\tilde{\mathcal{F}}$ of reduced local functionals inherits a Lie-structure from that on $\mathcal{F}_G$. In the sequel it is identified with $\mathcal{F}_G$.

Notice that by Proposition 5.1 the complex

$$\Omega_c^{0,0} \xrightarrow{d_H} \Omega_c^{1,0} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega_c^{n-1,0} \xrightarrow{d_H} \Omega_c^{n,0} \xrightarrow{\mathbf{E}} \Omega_c^{n,1} \to \cdots$$

induces a subcomplex

$$\Omega_\psi^{0,0} \xrightarrow{d_H} \Omega_\psi^{1,0} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega_\psi^{n-1,0} \xrightarrow{d_H} \Omega_\psi^{n,0} \xrightarrow{\mathbf{E}} \Omega_\psi^{n,1} \to \cdots$$

This subcomplex is exact up to the term $\Omega_\psi^{n-1,0}$ (and including it, i.e. $H_\psi^{n-1} = 0$), if we assume that $\Omega_c^{*,0}$ is itself exact and that every exact ψ-invariant form is the horizontal differential of some ψ-invariant form. Similarly, the subcomplex

$$\Omega_G^{0,0} \xrightarrow{d_H} \Omega_G^{1,0} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega_G^{n-1,0} \xrightarrow{d_H} \Omega_G^{n,0} \xrightarrow{\mathbf{E}} \Omega_G^{n,1} \to \cdots$$

is exact up to the term $\Omega_G^{n-1,0}$ (under the same assumptions).

6. An example: a Poisson sigma model

A number of authors [6,9,16] have investigated a class of physical theories called Poisson sigma models. These models focus on fields which are defined on a two-dimensional manifold Σ with range in a Poisson manifold M. These models seem to have first arisen in various theories of two-dimensional gravity but have been applied to areas such as topological field theory and in the reformulation of Kontsevich's work on deformation quantization [6]. We consider an application of our results to the version of the Poisson sigma model presented in the work of Ikeda [9] but we utilize the notation of [2].

Assume that V is a finite-dimensional vector space, say of dimension N and with basis $\{T_A\}$, and let $\{T^A\}$ denote the basis of the space V^* dual to V. We assume the existence of a Poisson tensor W on V^*. Thus W is a bivector field

$$W = W_{AB}\left(\frac{\partial}{\partial T_A} \wedge \frac{\partial}{\partial T_B}\right)$$

where, for each A, T_A is identified as a coordinate mapping $T_A : V^* \to \mathbf{R}$ and V is identified with V^{**}. The fact that W is Poisson means that it is a tensor and the $\{W_{AB}\}$ are functions on V^* (assumed to be polynomials in the coordinates $\{T_A\}$ in the present model) such that

$$W_{AD}\frac{\partial W_{BC}}{\partial T_D} + W_{BD}\frac{\partial W_{CA}}{\partial T_D} + W_{CD}\frac{\partial W_{AB}}{\partial T_D} = 0$$

and $W_{AB} = -W_{BA}$. Now V^* is a Poisson manifold with

$$\{f, g\} = W_{AB}\frac{\partial f}{\partial T_A}\frac{\partial g}{\partial T_B}$$

for smooth functions f, g defined on V^*.

Observe that the Poisson field W is not dependent on the basis used to represent it. If the components W_{AB} of W relative to a basis $\{T_A\}$ of V satisfy the Poisson conditions given above then the components $\overline{W}_{AB}$ of W relative to any other basis $\{\overline{T}_B\}$ of V will also satisfy these same conditions. Notice that $W_{AB} = \{T_A, T_B\}$ and that if $\{\overline{T}_A\}$ is a different basis and $\{\overline{W}_{AB}\}$ are the components of W relative to it then $\overline{W}_{AB} = \{\overline{T}_A, \overline{T}_B\}$ as well.

The fields of Ikeda's model are ordered pairs (ψ, h) where ψ is a mapping from the two-dimensional manifold Σ into V^* and h is a mapping from Σ into $T^*\Sigma \otimes V$. In components

$$\psi(x) = \psi_A(x)T^A, \quad h(x) = h^A_\mu(x)(\mathrm{d}x^\mu \otimes T_A)$$

where $\{x^\mu\}$ are coordinates on Σ. One form of Ikeda's Lagrangian for two-dimensional gravity is

$$L = \epsilon^{\mu\nu}\left[h^A_\mu D_\nu \psi_A - \frac{1}{2}W_{AB}h^A_\mu h^B_\nu\right],$$

where ϵ is the skew-symmetric Levi-Civita tensor on Σ such that $\epsilon^{01} = 1$ and

$$D_\nu \psi_A = \partial_\nu \psi_A + W_{AB} h^B_\nu.$$

It is our intent to show how some of our work relates to Ikeda's model. To cast this model in our formalism let E denote the vector bundle over Σ with total space $E = V^* \oplus [T^*\Sigma \otimes V]$ and with the obvious projection of E onto Σ. The fields (ψ, h) are sections of this bundle.

First we show how to define a Poisson bracket on the relevant space of local functionals. To accomplish this, we want to construct a mapping ω as in Section 2.1 in such a manner that the Jacobi condition is satisfied.

For this purpose we find it convenient to introduce a positive definite metric μ on V with its induced metric μ^* on V^*. Moreover we choose $\{T_A\}$ to be an orthonormal basis relative to μ and we define $\{T^B\}$ by $T^B(v) = \mu(v, T_B)$ for each B and for all $v \in V$. It follows that $\{T^A\}$ is a μ^*-orthonormal basis of V^* and that the basis $\{T^A\}$ is dual to $\{T_B\}$. Define a tensor $\tilde{W}$ on V by

$$\tilde{W} = W^{AB} \left(\frac{\partial}{\partial T^A} \wedge \frac{\partial}{\partial T^B} \right)$$

where $W^{AB} = \mu^{AC}\mu^{BD}W_{CD}$, and $\mu^{PQ} = \mu^*(T^P, T^Q)$ for $1 \le P, Q \le N$. Thus $\tilde{W}$ is the tensor on V induced by W and the metric μ.

We reformulate this data in terms of the jet bundle of E. In particular the tensors $w, \tilde{w}$ induce a bilinear mapping ω on the jet bundle which is used to define a Lie structure on the space of functionals. Local coordinates on E may be denoted (x^μ, u_A, w^B_μ) and those on the jet bundle $J^\infty E$ by $(x^\mu, u_{A_I}, w^B_{\mu,J})$. Thus if (ψ, h) is a section of E we have

$$x^\mu((\psi, h)(p)) = x^\mu(p), \qquad u_A((\psi, h)(p)) = \psi_A(p)$$

and

$$w^B_\mu((\psi, h)(p)) = h^B_\mu.$$

Clearly, there is a corresponding splitting of the jet coordinates. It follows that in local coordinates each local function P on $J^\infty E$ is a function of $(x^\mu, u_{A,I}, w^B_{\nu,J})$. Now the $\{W_{AB}\}$ are functions of the coordinates $\{T_A\}$ on V and these coordinates are denoted $\{u_A\}$ on the bundle E. Consequently we can regard the $\{W_{AB}\}$ as being functions on the jet bundle $J^\infty E$ which depend polynomially on the coordinates $\{u_A\}$ and are in fact independent of the coordinates $x^\mu, u_{A,I}, w^B_{\mu,J}$ for $|I| \ge 1$. The function ω is required to be a mapping from $\Omega^{n,1}_0 \times \Omega^{n,1}_0$ into Loc_E. Observe that there are two types of contact forms θ_A, θ^B_μ on $J^\infty E$, those which arise from the coordinates $\{u_A\}$ and those which arise from $\{w^B_\mu\}$. Since each fiber of E is a direct sum of two vector spaces the matrix of components of ω is a block diagonal matrix with two blocks defined by

$$\omega_{AB} = W_{AB} \quad \text{and} \quad \omega^{A,B}_{\mu,\nu} = \delta_{\mu\nu} W^{AB}.$$

Here $\delta_{\mu\nu}$ is the usual Kronecker delta symbol. In the first block we have written the indices of ω^{ab} as lower indices as they represent components relative to a basis of the dual of V. In the second block it is appropriate to write the components $\omega^{(\mu,A),(\nu,B)}$ as $\omega^{A,B}_{\mu,\nu}$ for similar reasons. Notice that the matrix of ω is skew-symmetric and

$$\omega_{AD}\frac{\partial \omega_{BC}}{\partial u_D} + \omega_{BD}\frac{\partial \omega_{CA}}{\partial u_D} + \omega_{CD}\frac{\partial \omega_{AB}}{\partial u_D} = 0.$$

The other components of ω satisfy a similar condition as they too are determined by the $\{W_{AB}\}$. It follows from this fact and Eq. (7.11) of [15] that the bracket of local functionals defined on sections of E by

$$\{\mathcal{P}, \mathcal{Q}\}(\phi) = \int_\Sigma [\omega^{ab}\mathbf{E}_a(P)\mathbf{E}_b(Q)] \circ (j^\infty \phi)\nu$$

satisfies the Jacobi identity (see the discussion in Section 2.1).

Ikeda shows that the Euler operators for the Lagrangian L in this model are given by

$$\mathbf{E}^A(L) = \epsilon^{\mu\nu} R^A_{\mu\nu}, \quad \mathbf{E}^\mu_A(L) = \epsilon^{\mu\nu} D_\nu \psi_A$$

where

$$R^A_{\mu\nu} = \partial_\mu h^A_\nu - \partial_\nu h^A_\mu + \frac{\partial W_{BC}}{\partial T_A} h^B_\mu h^C_\nu.$$

This suggests that for every local function P we should define

$$\mathbf{E}^A(P) = (-D)_I \left(\frac{\partial P}{\partial u_{A,I}}\right), \qquad \mathbf{E}^\mu_B(P) = (-D)_J \left(\frac{\partial P}{\partial w^B_{\mu,J}}\right).$$

Consequently, the Poisson bracket assumes the following form:

$$\{\mathcal{P}, \mathcal{Q}\}(\psi, h) = \int_\Sigma [\omega_{AB}\mathbf{E}^A(P)\mathbf{E}^B(Q)] \circ j^\infty(\psi, h)\nu + \int_\Sigma [\omega^{A,B}_{\mu,\nu}\mathbf{E}^\mu_A(P)\mathbf{E}^\nu_B(Q)] \circ j^\infty(\psi, h)\nu.$$

Now we characterize the automorphisms of E that induce canonical transformations, relative to the Poisson bracket, on the space of functionals $\mathcal{F}$. Recall that in Section 2.2 we have referred to such automorphisms as canonical automorphisms. Suppose that Ψ is a linear automorphism of E, i.e., assume that there are matrices R, S which are functions on Σ such that

$$\Psi([T^A \oplus (\mathrm{d}x^\mu \otimes T_B)]) = (R^A_C T^C) \oplus (\mathrm{d}x^\mu \otimes (S^D_B T_D)).$$

We determine conditions which insure that Ψ is a canonical automorphism of E. According to Lemma 2.1 of Section 2.1 such an automorphism will be canonical iff its components satisfy condition (ii) of the Lemma.

Observe that $\Psi_A = u_A \circ \Psi = R^C_A u_C$, $\Psi^A_\mu = w^A_\mu \circ \Psi = S^A_D w^D_\mu$ and

$$\frac{\partial \Psi_A}{\partial u_B} = R^B_A, \qquad \frac{\partial \Psi^A_\mu}{\partial w^B_\nu} = S^A_B \delta^\nu_\mu.$$

Consequently, if the matrices satisfy the conditions

$$\bar{W}_{AB} = W_{CD} R^C_A R^D_B, \qquad \bar{W}^{AB} = W^{CD} S^A_C S^B_D$$

where $\bar{W}_{AB}$ are the components of the tensor W relative to a new basis $\bar{T}_A = M^C_A T_C$ and $\bar{T}^A = (M^{-1})^A_D T^D$, then

$$\tilde{\omega}_{AB} = \omega(\tilde{\theta}_A, \tilde{\theta}_B) = \bar{W}_{AB} = W_{CD} R^C_A R^D_B = \omega(\theta_C, \theta_D) \frac{\partial \Psi_A}{\partial u_C} \frac{\partial \Psi_B}{\partial u_D}$$

where, we have dropped the volume form ν from the definition of the components of ω, as they were defined in Section 2.1, for simplicity. Similarly, we must have

$$\tilde{\omega}^{A,B}_{\mu,\nu} = \omega(\tilde{\theta}^A_\mu, \tilde{\theta}^B_\nu) = \delta_{\mu\nu} \bar{W}^{AB} = W^{CD} \frac{\partial \Psi^A_\mu}{\partial w^C_\lambda} \frac{\partial \Psi^B_\nu}{\partial w^D_\rho} \delta_{\lambda\rho}$$

which is the same as

$$\tilde{\omega}^{A,B}_{\mu,\nu} = \omega^{C,D}_{\lambda,\rho} \frac{\partial \Psi^A_\mu}{\partial w^C_\lambda} \frac{\partial \Psi^B_\nu}{\partial w^D_\rho}.$$

Notice that these computations will be consistent if we require that $R = M$ and $S = M^{-1}$ since M is the matrix transforming the basis $\{T_A\}$ to $\{\bar{T}_A\}$, and since we require that W and $\tilde{W}$ be tensors. Moreover we must also have that M be orthogonal if we want the transformed basis to remain μ-orthonormal. These remarks give us the conditions required in order that a linear automorphism be canonical.

If G is a Lie group and $M : G \to O(n)$ is a *representation* of G by orthogonal matrices then there is a representation Φ of G via canonical automorphisms of $E = V^* \oplus (T^*\Sigma \otimes V)$ defined by

$$\Phi(g)([T^A \oplus (\mathrm{d}x^\mu \otimes T_B)]) = (M(g)^A_C T^C) \oplus (\mathrm{d}x^\mu \otimes ([M(g)^{-1}]^D_B T_D)).$$

The fact that $\Phi : G \to Aut(E)$ is a *group homomorphism* is a consequence of the fact that M defines a linear left action of G on V via

$$g \cdot T_A = [M(g)^{-1}]_A^B T_B$$

with a corresponding linear left action of G on V^* defined by

$$g \cdot T^A = M(g)_B^A T^B.$$

The following theorem is a consequence of these remarks:

Theorem 6.1. *For each orthogonal $n \times n$ matrix M there is a canonical gauge automorphism Ψ_M of the bundle $V^* \oplus (T^*\Sigma \otimes V) \longrightarrow \Sigma$ which is linear on fibers of the bundle and which transforms the basis $\{T^A \oplus (\mathrm{d}x^\mu \otimes T_B)\}$ via*

$$\Psi_M([T^A \oplus (\mathrm{d}x^\mu \otimes T_B)]) = (M_C^A T^C) \oplus (\mathrm{d}x^\mu \otimes ((M^{-1})_B^D T_D)).$$

Moreover, if $M : G \to O(n)$ is a representation of a Lie group G by orthogonal $n \times n$ matrices, then the mapping $\Phi : G \to Aut(V^* \oplus (T^*\Sigma \otimes V))$ defined by $\Phi(g) = \Psi_{M(g)}$ for $g \in G$, is a representation of G by canonical automorphisms of $V^* \oplus (T^*\Sigma \otimes V)$. It follows that the space of local functionals defined on sections of the bundle $V^* \oplus (T^*\Sigma \otimes V) \to \Sigma$ admits a reduction as does the complex $\{\Omega_c^{k,0}[J^\infty(V^* \oplus (T^*\Sigma \otimes V))], k = 0, 1, 2, \ldots, n\}$.

Remark. It is not difficult to show that Ikeda's Lagrangian given above is invariant under the action of the Lie group G defined in Theorem 6.1.

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